SPOT INVERSION IN THE HESTON MODEL

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Abstract. We analyse the Heston stochastic volatility model under an inversion of spot. The result is that under the appropriate measure changes the resulting process is again a Heston type process whose parameters can be explicitly determined from those of the original process. This behaviour can be interpreted as some measure of *sanity* of the Heston model but does not seem to be a general feature of stochastic volatility processes.

This note is concerned with an observation related to the inversion of spot in the Heston model. Although not immediately of interest in equity or rates modelling, the inverse of spot is a very natural process in foreign exchange modelling where it is simply the exchange rate for the reverse currency pair. Therefore a sanity measure for a model is given by how well it fares under a spot inversion. This test is easily passed by the log-normal Black-Scholes model but it is less clear for general stochastic volatility dynamics; for example it is not obvious how a stochastic volatility model with log-normal volatility behaves with respect to spot inversion. In this note we will show that the Heston displays a somewhat remarkable symmetry with respect to spot inversion.

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1. The Heston model

1.1. **The Heston dynamics.** The Heston model ([H]) is a stochastic volatility model in which the instantaneous variance follows a mean

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reverting square root process

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{\mathrm{spot}}$$
$$\mathrm{d}V_t = \kappa(\theta - V_t)\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{\mathrm{vol}}.$$

Where $\mathrm{d}W_t^{\mathrm{spot}}$ and $\mathrm{d}W_t^{\mathrm{vol}}$ are two jointly Brownian motions with constant correlation ρ and μ, κ, θ and ω are constants.

The spot process is understood to be the number of units of *term* currency worth one unit of *base* currency. For example, for EURUSD it is the U.S. Dollar (term) worth of one Euro (base). Spot is therefore quoted in the term currency. If we use the term rolling cash account as numéraire then the drift becomes the interest rate differential $(r_{\mathrm{term}} - r_{\mathrm{base}})$. Since we will need to keep track of the numéraire we will add a superscript to indicate the measure for our Brownian motions.

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= (r_{\mathrm{term}} - r_{\mathrm{base}})\,\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{\mathrm{spot,\,term}} \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{\mathrm{vol,\,term}}. \end{aligned} \tag{1}$$

1.2. **Pricing of vanillas in the Heston model.** Thanks to the work of Steven Heston ([H]) we know how to price vanilla options in the model described above. The basic idea is to find the characteristic function of spot analytically and then retrieve the price of an option using the fact that the Fourier transform is an isometry. If we denote the discount factors to delivery date T by Df_{term} and Df_{base} and the forward by $F_T = S_0 \cdot Df_{\mathrm{base}}/Df_{\mathrm{term}}$; then following the notation of [JK] the price in term currency of a base currency call / term currency put struck at K in size $Notional_{\mathrm{base}}$ is

$$Df_{\mathrm{term}} \cdot Notional_{\mathrm{base}} \cdot \left[\frac{1}{2}(F_T - K) + \frac{1}{\pi}\int_0^\infty (F_T \cdot f_1 - K \cdot f_2)\mathrm{d}u\right] \tag{2}$$

where f_1 and f_2 are defined by

$$f_1 := \Re\left(\frac{\mathrm{e}^{-iu\ln K}\phi(u-i)}{iuF_T}\right) \qquad \textit{and} \qquad f_2 := \Re\left(\frac{\mathrm{e}^{-iu\ln K}\phi(u)}{iu}\right)$$

and ϕ is the characteristic function of log-spot to expiry

$$\phi(u) = \mathbb{E}\left(\mathrm{e}^{iu\ln(S_T)}\right)$$

which in the Heston model is given by

$$\phi(u) = \mathrm{e}^{C(T,u)+D(T,u)V_0+iu\ln(F_T)}$$

where

$$C(T,u) = \frac{\kappa\theta}{\omega^2}\left((\kappa - \rho\omega u i + d(u))T - 2\ln\left(\frac{c(u)\mathrm{e}^{d(u)T} - 1}{c(u) - 1}\right)\right)$$

$$D(T,u) = \frac{\kappa - \rho\omega u i + d(u)}{\omega^2}\left(\frac{\mathrm{e}^{d(u)T} - 1}{c(u)\mathrm{e}^{d(u)T} - 1}\right)$$

and

$$c(u) = \frac{\kappa - \rho\omega u i + d(u)}{\kappa - \rho\omega u i - d(u)} \qquad d(u) = \sqrt{(\rho\omega u i - \kappa)^2 + iu\omega^2 + \omega^2 u^2}\,.$$

2. Inversion of the Heston model

2.1. **Heuristic inversion of the spot process.** In this section we shall study the dynamics for inverse spot in a Heston model. When changing numéraire to the foreign numéraire there is a degree of ambiguity as to what to do with the volatility Brownian driver. We will heuristically make a choice which surprisingly gives rise to a Heston model, then in the next section we prove that this choice prices vanillas consistently with the original model and so is the Heston model that matches the market.

To invert the spot process we use the Ito formula

$$\frac{\mathrm{d}S_t^{-1}}{S_t^{-1}} = (r_{\text{base}} - r_{\text{term}} + V_t)\mathrm{d}t - \sqrt{V_t}\mathrm{d}W_t^{\text{spot, term}}.$$

With this Brownian motion it is false that tradables are martingales since for example the expectation of (inverse) spot is not the (inverse) forward (this is called Siegel's paradox). To remedy this we need to change the measure, it is easy to see that the appropriate change is

$$\mathrm{d}W_t^{\text{spot, base}} = -\mathrm{d}W_t^{\text{spot, term}} + \sqrt{V_t}\mathrm{d}t. \tag{3}$$

Then (appropriately discounted) inverse spot becomes a martingale

$$\frac{\mathrm{d}S_t^{-1}}{S_t^{-1}} = (r_{\text{base}} - r_{\text{term}})\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{\text{spot, base}}.$$

In the Black Scholes framework (ie. V_t deterministic) this is the end of the story regarding spot inversions. However in Heston the Brownian process that drives the variance process is correlated with the spot process

$$\mathrm{d}W_t^{\text{vol,term}} = \rho\mathrm{d}W_t^{\text{spot, term}} + \overline{\rho}\mathrm{d}\overline{W}_t^{\text{term}}$$

where as usual $\overline{\rho} = \sqrt{1-\rho^2}$ and $\overline{W}_t^{\text{term}}$ is a Brownian motion independent of $W_t^{\text{spot, term}}$. This means that the measure change in (3) will affect the drift of the instantaneous variance process by an amount proportional to the correlation. If we somehow arbitrarily assume that

that is the only change of measure required and leave $d\overline{W}_t$ components to be the same by defining (this is the heuristic part of the argument)

$$\mathrm{d}\overline{W}_t^{\mathrm{base}} := \mathrm{d}\overline{W}_t^{\mathrm{term}}$$

then we can write down the change of measure on the Brownian driving volatility

$$\begin{aligned}
\mathrm{d}W_t^{\mathrm{vol,\,term}} &= \rho \mathrm{d}W_t^{\mathrm{spot,\,term}} + \overline{\rho}\mathrm{d}\overline{W}_t^{\mathrm{term}} \\
&= \rho\left(-\mathrm{d}W_t^{\mathrm{spot,\,base}} + \sqrt{V_t}\mathrm{d}t\right) + \overline{\rho}\mathrm{d}\overline{W}_t^{\mathrm{base}} \\
&= \rho\sqrt{V_t}\mathrm{d}t + \mathrm{d}W_t^{\mathrm{vol,\,base}}
\end{aligned}$$

where $\mathrm{d}W_t^{\mathrm{vol,\,base}}$ is defined by the equation above. Then we have that inverse spot follows the following stochastic volatility process

$$\begin{aligned}
\frac{\mathrm{d}S_t^{-1}}{S_t^{-1}} &= (r_{\mathrm{base}} - r_{\mathrm{term}})\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{\mathrm{spot,\,base}} \\
\mathrm{d}V_t &= \kappa_n(\theta_n - V_t)\mathrm{d}t + \omega\sqrt{V_t}\mathrm{d}W_t^{\mathrm{vol,\,base}}
\end{aligned}$$

where $W_t^{\mathrm{spot,\,base}}$ and $W_t^{\mathrm{vol,\,base}}$ are jointly normal with correlation ρ_n and

$$\begin{aligned}
\rho_n &= -\rho \\
\kappa_n &= \kappa - \rho\omega \\
\theta_n &= \theta\kappa_n/\kappa.
\end{aligned} \tag{4}$$

It would be rather optimistic to expect this change of parameters to price options consistently with the Heston model (1); we will show this to be indeed the case in the next section.

2.2. **Pricing in the reverse currency.** This section contains the main result of this note which is that the price of an option in the Heston model (1) is the same as in the Heston model for the reciprocal currency pair with the parameters modified as in (4).

Take a base call/term put struck at K in notional N_{base}, according to (2) the price of such an option is

$$Df_{\mathrm{term}} \cdot N_{\mathrm{base}} \cdot \left(\frac{1}{2}(F_T - K) + \frac{1}{\pi}\int_0^\infty (F_T \cdot f_1 - K \cdot f_2)\mathrm{d}u\right). \tag{5}$$

By put-call parity the price of an equivalent base put/term call option is obtained by simply reversing the sign in front of $1/\pi$.

The transaction above is viewed by the "inverse" investor as a term put / base call struck at $1/K$ in notional $N_{term} = N_{\mathrm{base}} \cdot K$, the pricing formula (2) (adapted for put options) states the price of such an option is

(6)

$$Df_{\mathrm{base}} \cdot S_0 \cdot N_{\mathrm{term}} \cdot \left(\frac{1}{2}(F_T^{-1} - K^{-1}) - \frac{1}{\pi}\int_0^\infty (F_T^{-1} \cdot f_1^n - K^{-1} \cdot f_2^n)\mathrm{d}u\right)$$

where we have added a spot conversion to yield the price in term currency.

The main result of this note is

Theorem 1. *The price in (5) is equal to (6) provided the parameters for the reverse spot process in (6) are chosen to be the ones in (4).*

In the proof we will freely add the subscript "n" to indicate we are using the parameters in (4) with the reverse spot process. The Heston formulæ above use the characteristic function of log-spot, the following lemma clarifies the connection between the characteristic function for spot and reverse spot.

Lemma 1. *We have* $\phi_n(u) = F_T^{-1}\phi(-u-i)$

Proof. As explained in section 1.2 the characteristic function ϕ is defined in terms of the functions $C(T,u)$ and $D(T,u)$ which are in turn described in terms of the functions $d(u)$ and $c(u)$ defined therein. To connect the functions we simply observe

$$
\begin{aligned}
d_n(u)^2 &= (\rho_n\omega ui - \kappa_n)^2 + iu\omega^2 + \omega^2u^2 \\
&= (-\rho\omega ui - \kappa + \rho\omega)^2 + \omega^2 u(u+i) \\
&= (\rho\omega(-u-i)i - \kappa + \rho\omega)^2 + \omega^2 + \omega^2u^2 \\
&= d(-u-i)^2. \qquad (7)
\end{aligned}
$$

This implies $d_n(u) = \pm d(-u-i)$. Assume for the moment that $d_n(u) = d(-u-i)$ then

$$
\begin{aligned}
c_n(u) &= \frac{\kappa_n - \rho_n\omega ui + d_n(u)}{\kappa_n - \rho_n\omega ui - d_n(u)} \\
&= \frac{\kappa - \rho\omega + \rho\omega ui + d(-u-i)}{\kappa - \rho\omega + \rho\omega ui - d(-u-i)} \\
&= \frac{\kappa - \rho\omega(-u-i)i + d(-u-i)}{\kappa - \rho\omega(-u-i)i - d(-u-i)} \\
&= c(-u-i)
\end{aligned}
$$

from this it follows that

$$C_n(T,u) = C(T,-u-i)$$

and

$$D_n(T,u) = D(T,-u-i)$$

which yields the characteristic function for reverse spot

$$\begin{aligned}\phi_n(u) &= \mathrm{e}^{C_n(T,u)+D_n(T,u)V_0+iu\ln(F_T^{-1})} \\ &= \mathrm{e}^{C(T,-u-i)+D_n(T,-u-i)V_0-iu\ln(F_T)} \\ &= \mathrm{e}^{C(T,-u-i)+D_n(T,-u-i)V_0+i(-u-i)\ln(F_T)}F_T^{-1} \\ &= F_T^{-1}\phi(-u-i)\end{aligned}$$

as claimed.

If above we had assumed that $d_n(u) = -d(-u-i)$ then it is easy to see that $c_n(u) = 1/c(-u-i)$. From this we can derive $C_n(T,u) = C(T,-u-i)$ and $D_n(T,u) = D(T,-u-i)$ again from which the result follows. □

Proof of Theorem. In order to verify the agreement of (5) and (6):

$$\begin{aligned} &Df_{\text{term}} \cdot N_{\text{base}} \cdot \left(\frac{1}{2}(F_T - K) + \frac{1}{\pi}\int_0^\infty (F_T\cdot f_1 - K\cdot f_2)\,\mathrm{d}u\right) = \\ &= Df_{\text{base}} \cdot S_0 \cdot N_{\text{term}} \cdot \\ &\qquad \left(\frac{1}{2}\left(F_T^{-1} - K^{-1}\right) - \frac{1}{\pi}\int_0^\infty \left(F_T^{-1}\cdot f_1^n - K^{-1}\cdot f_2^n\right)\mathrm{d}u\right)\end{aligned}$$

we divide by the discounted notional which yields

$$\begin{aligned} &\frac{1}{2}(F_T - K) + \frac{1}{\pi}\int_0^\infty (F_T\cdot f_1 - K\cdot f_2)\,\mathrm{d}u = \\ &= F_T\cdot K\cdot\left(\frac{1}{2}\left(F_T^{-1} - K^{-1}\right) - \frac{1}{\pi}\int_0^\infty \left(F_T^{-1}\cdot f_1^n - K^{-1}\cdot f_2^n\right)\mathrm{d}u\right)\end{aligned}$$

that is, we need to show the equality

$$F_T - K = \frac{1}{\pi}\int_0^\infty (F_T\cdot f_1 - K\cdot f_2)\,\mathrm{d}u - \frac{1}{\pi}\int_0^\infty (K\cdot f_1^n - F_T\cdot f_2^n)\,\mathrm{d}u.$$

To prove this we expand the functions under the integral

$$\begin{aligned} &\frac{1}{\pi}\int_0^\infty \left(F_T\cdot\Re\left(\frac{\mathrm{e}^{-ui\ln K}\phi(u-i)}{iuF_T}\right) - K\cdot\Re\left(\frac{\mathrm{e}^{-ui\ln K}\phi(u)}{iu}\right)\right)\mathrm{d}u \\ &\quad - \frac{1}{\pi}\int_0^\infty \left(K\cdot\Re\left(\frac{\mathrm{e}^{-ui\ln K^{-1}}\phi^n(u-i)}{iuF_T^{-1}}\right)\right. \\ &\qquad\qquad \left. - F_T\cdot\Re\left(\frac{\mathrm{e}^{-ui\ln K^{-1}}\phi^n(u)}{iu}\right)\right)\mathrm{d}u\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^\infty \Bigg(F_T \cdot \Re\left(\frac{\mathrm{e}^{-ui\ln K}\phi(u-i)}{iuF_T}\right) \\
&\qquad\qquad - K \cdot \Re\left(\frac{\mathrm{e}^{-ui\ln K}\phi(u)}{iu}\right)\Bigg)\mathrm{d}u \\
&\quad - \frac{1}{\pi}\int_0^\infty \Bigg(K \cdot \Re\left(\frac{\mathrm{e}^{ui\ln K}F_T^{-1}\phi(-u+i-i)}{iuF_T^{-1}}\right) \\
&\qquad\qquad - F_T \cdot \Re\left(\frac{\mathrm{e}^{ui\ln K}F_T^{-1}\phi(-u-i)}{iu}\right)\Bigg)\mathrm{d}u \\
&= \frac{1}{\pi}\int_0^\infty \Bigg(F_T \cdot \Re\left(\frac{\mathrm{e}^{-ui\ln K}\phi(u-i)}{iuF_T}\right) \\
&\qquad\qquad - K \cdot \Re\left(\frac{\mathrm{e}^{-ui\ln K}\phi(u)}{iu}\right)\Bigg)\mathrm{d}u \\
&\quad - \frac{1}{\pi}\int_0^\infty \Bigg(K \cdot \Re\left(\frac{\mathrm{e}^{ui\ln K}\phi(-u+i-i)}{iu}\right) \\
&\qquad\qquad - F_T \cdot \Re\left(\frac{\mathrm{e}^{ui\ln K}\phi(-u-i)}{iuF_T}\right)\Bigg)\mathrm{d}u.
\end{aligned}
$$

By the Cauchy residue theorem this is equal to

$$
\begin{aligned}
&\Re\left(F_T \cdot \mathrm{Res}_{u=0}\frac{\mathrm{e}^{-iu\ln K}\phi(u-i)}{iuF_T} - K\cdot \mathrm{Res}_{u=0}\frac{\mathrm{e}^{-iu\ln K}\phi(u)}{iu}\right) \\
&\quad = \Re\left(F_T \cdot \frac{\phi(-i)}{iF_T} - K \cdot \frac{\phi(0)}{iu}\right) \\
&\quad = F_T - K.
\end{aligned}
$$

This proves the theorem. □

3. Application to Variance Swaps

A variance swap is a contract that pays out a linear function of the realised historical variance of the returns of an asset in a specified set of dates. An example of a fairly standard deal is a one year USDJPY variance swap paying USD100k per volatility point where USDJPY is taken every business day at 4pm London time from a given Reuters fixing page. A deal like this is quoted by giving a fair variance level in a similar fashion to how forwards and futures are dealt.

Obviously the volatility exposure of a USDJPY variance swap paying a rebate in USD is different from that paying an equivalent amount of JPY. The reason for this is that USDJPY spot is negatively correlated with its volatility. A variance swap paying in JPY is more valuable since a high realised variance scenario is likely to occur on JPY gaining value against USD. Inexperienced dealers can be arbitraged in this way.

In mathematical modelling the variance is replaced by the continuously sampled variance which is also called quadratic variation, in the Heston model this is the stochastic variable

$$\frac{1}{T}\int_0^T V_t \mathrm{d}t$$

where T is the expiration date of the contract expressed in years. The fair level for a variance swap paying in term currency is the expectation of the annualised accrued variance which in the Heston model is simply

$$\mathbb{E}_{term}\left(\frac{1}{T}\int_0^T V_t \mathrm{d}t\right) = \theta + (V_0 - \theta)\frac{1-\mathrm{e}^{-\kappa T}}{\kappa T}$$

which is a number between the starting level of variance V_0 and its mean reversion level θ.

For a variance swap paying in base currency we just need to invert the Heston process and apply this formula, the level is therefore

$$\mathbb{E}_{base}\left(\frac{1}{T}\int_0^T V_t \mathrm{d}t\right) = \theta\frac{\kappa}{\kappa-\rho\omega} + \left(V_0 - \theta\frac{\kappa}{\kappa-\rho\omega}\right)\frac{1-\mathrm{e}^{-(\kappa-\rho\omega)T}}{(\kappa-\rho\omega)T}$$

which will be a number between the starting level V_0 and the modified mean reversion $\theta\kappa/(\kappa-\rho\omega)$. These calculations are in agreement with the observations above regarding USDJPY since in this case the correlation ρ is negative which depresses the value of the variance swap level when pricing in base currency.

References

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